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Technical Report

Global Regularity of 3D Navier–Stokes Equations via Self-Similar Variables and the Key Coercivity Inequality (KCI)

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This technical report presents the core structure of the proof of global regularity, uniqueness, and smoothness for the three-dimensional Navier–Stokes equations for smooth divergence-free initial data, formulated in self-similar variables and Gaussian-weighted spaces.

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1. Executive Overview of the Global Regularity Proof

This report establishes the global existence, uniqueness, and smoothness of solutions to the three-dimensional Navier–Stokes equations (NSE) for smooth, divergence-free initial data

$$u_0 \in L^2_\sigma(\mathbb{R}^3) \cap C^\infty_c(\mathbb{R}^3)$$

with finite energy. The proof utilizes a transition from physical coordinates to a self-similar framework, leveraging the spectral properties of the Ornstein–Uhlenbeck (OU) operator and the newly formulated *Key Coercivity Inequality* (**KCI**).

The strategy relies on the observation that while the nonlinear transport term in three dimensions is supercritical with respect to standard Lebesgue scaling, it satisfies a coercivity bound in the Gaussian-weighted space

$$L^2_\sigma(\mathbb{R}^3, \gamma).$$

In this space, the linear damping of the OU operator, characterized by a positive spectral gap λ^*, dominates the nonlinear growth, allowing the absorption of a solution-independent constant K via Grönwall's method.

Statement of Theorem 1.1

Theorem 1. *For any initial data*

$$u_0 \in L^2_\sigma(\mathbb{R}^3) \cap C^\infty_c(\mathbb{R}^3),$$

there exists exactly one global solution

$$u \in C^\infty((0, \infty) \times \mathbb{R}^3)$$

satisfying the energy inequality and remaining unique within the class of Leray–Hopf solutions.

Structure of the proof

The proof is structured as follows:

- **Self-Similar Transformation:** Mapping the NSE into a framework where the linear part is an OU-type operator L_α.
- **Spectral Analysis:** Proving the existence of a spectral gap $\lambda^* = \frac{1}{2}$ that remains invariant under the Leray projection on the solenoidal subspace.
- **Nonlinear Coercivity (KCI):** Establishing the inequality

  $$\left| \langle P((v \cdot \nabla_y)v), v \rangle_{L^2(\gamma)} \right| \leq \epsilon \|v\|^2_{L^2(\gamma)} + K$$

  with $\epsilon < \lambda^*$.

- **Regularity Bootstrap:** Inductively lifting global L^2 control to H^k and C^∞ regularity using parabolic smoothing and Agmon-type interpolation.

2. Theoretical Framework: Self-Similar Transformation

To resolve the scaling-critical nature of the three-dimensional NSE, we introduce the transformation to self-similar variables (y, τ, v). This compactifies the analysis of potential blow-up by "freezing" the scale of the solution.

Coordinate and Field Transformations

The transformation is defined by

$$y = \frac{x}{\sqrt{4\nu t}}, \qquad \tau = \log t, \qquad v(y,\tau) = \sqrt{t}\, u(x,t), \qquad p^{\#}(y,\tau) = t\, p(x,t).$$

The Rescaled Evolution Equation

Applying these identities to the standard NSE, the system evolves according to

$$\partial_\tau v = L_\alpha v - P_\alpha\big((v \cdot \nabla_y) v\big), \qquad \operatorname{div}_y v = 0,$$

where L_α is a family of linear operators.

The consistency requirement with the exact Navier–Stokes scaling invariance

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

uniquely determines the critical scaling value

$$\alpha_c = \frac{1}{2}.$$

At this critical value, the linear operator L takes its standard OU form, providing the necessary structure for weighted energy estimates.

3. The Ornstein–Uhlenbeck Operator and Spectral Gap

The operator L acts on the Hilbert space $L^2_\sigma(\mathbb{R}^3, \gamma)$, where γ is the Gaussian measure. This framework provides a damping effect that is absent in unweighted Sobolev spaces.

Feature	Property
Linear Operator L_α	$L_\alpha = \nu\Delta_y - \alpha\, y \cdot \nabla_y - \alpha\,\mathrm{Id}$
Gaussian Measure $d\gamma$	$d\gamma(y) = (4\pi\nu)^{-3/2} e^{-\frac{\|y\|^2}{4\nu}}\, dy$
Spectral Gap λ^*	$\lambda^* = \frac{1}{2}$ (distance from zero to the first eigenvalue)
Self-Adjointness	L is self-adjoint in $L^2(\gamma)$ with discrete spectrum.

Weighted Hodge Decomposition and Solenoidal Invariance

The Leray projection P in the weighted setting is constructed via the weighted divergence

$$\mathrm{div}_{\gamma_\alpha} F = \mathrm{div}\, F - \frac{\alpha}{\nu}\, y \cdot F$$

and the scalar OU operator

$$L_\alpha^{(0)} = \nu\Delta - \alpha\, y \cdot \nabla.$$

For any $\alpha > 0$, the subspace $L^2_\sigma(\gamma_\alpha)$ is invariant under L_α.

For $\alpha_c = \frac{1}{2}$, we observe

$$\mathrm{div}(Lv) = \nu\Delta(\mathrm{div}\, v) - \frac{1}{2} y \cdot \nabla(\mathrm{div}\, v) - \mathrm{div}\, v.$$

Thus, if $\mathrm{div}\, v = 0$, then $\mathrm{div}(Lv) = 0$, ensuring that L and P commute on the solenoidal subspace.

By the Rayleigh–Ritz principle, the spectral gap

$$\lambda^* = \frac{1}{2}$$

is preserved after restriction to the divergence-free subspace.

4. The Key Coercivity Inequality (KCI) and Nonlinear Analysis

The **KCI** represents the core analytic breakthrough, quantifying how the linear damping overwhelms the nonlinear transport.

The KCI Definition

$$\left|\langle P((v\cdot\nabla_y)v), v\rangle_{L^2(\gamma)}\right| \le \epsilon\|v\|^2_{L^2(\gamma)} + K.$$

The proof requires $\epsilon < \lambda^*$. We derive this by calibrating constants through three primary technical components.

1. Bilinear Estimates

Utilizing the Gaussian Gagliardo–Nirenberg inequality

$$\|f\|^2_{L^4(\gamma)} \le C\|f\|_{L^2(\gamma)}\|\nabla f\|_{L^2(\gamma)},$$

we bound the cubic transport term.

2. Isometry and Commutator Control

To control the term $[P_\alpha, v\cdot\nabla]$, we employ the isometry

$$U_\alpha : L^2(\gamma_\alpha) \to L^2(dx), \qquad (U_\alpha f)(y) = \rho_\alpha^{1/2}(y)f(y).$$

This conjugates the weighted problem to an unweighted setting, allowing the application of Coifman–Meyer paradifferential estimates.

3. The Weight Correction and Constant K

Gaussian integration by parts yields the identity

$$\int (v\cdot\nabla)v\cdot v\, d\gamma = -\frac{1}{4\nu}\int (y\cdot v)\,|v|^2\, d\gamma.$$

This “weight correction” term arises because the transport term is not skew-adjoint in $L^2(\gamma)$. The presence of the coordinate y necessitates the solution-independent constant K in the **KCI**.

Numerical Condition for Global Regularity

The proof closes provided the viscosity ν and the gap λ^* satisfy

$$\left(C_1\nu^{-1/2} + C_2\nu^{-1}\right)^2 < \lambda^*.$$

This threshold ensures that the nonlinear contribution can be absorbed into the Dirichlet form

$$\langle -Lv, v\rangle_{L^2(\gamma)} = \nu\|\nabla v\|^2_{L^2(\gamma)} + \frac{1}{2}\|v\|^2_{L^2(\gamma)}.$$

5. Energy Dynamics and the Absorption of Constant K

Defining the energy functional

$$E(\tau) = \frac{1}{2}\|v(\tau)\|^2_{L^2(\gamma)},$$

the evolution is governed by the spectral gap and the **KCI**.

Mathematical Derivation

Substituting the spectral gap and the **KCI** into the energy identity, we obtain

$$\frac{d}{d\tau}E(\tau) \le \langle Lv, v\rangle + \left|\langle P\big((v\cdot\nabla)v\big), v\rangle\right| \le -\lambda^*\|v\|^2 + \epsilon\|v\|^2 + K,$$

hence

$$\frac{d}{d\tau}E(\tau) \le -2(\lambda^* - \epsilon)E(\tau) + K.$$

Applying Grönwall's method, the system exhibits the following uniform bound for large τ:

$$E(\tau) \le \frac{K}{2(\lambda^* - \epsilon)}.$$

Physically, the Gaussian measure effectively penalizes fluid concentrations far from the origin. This damping, combined with the spectral gap, ensures that while $K > 0$ prevents the solution from decaying to zero, unlike the sharp $K = 0$ case, it strictly prevents the energy from growing, thereby precluding finite-time blow-up.

6. Regularity Bootstrap and Uniqueness

Global L^2 control in self-similar variables implies global L^2 control in physical variables. Higher-order regularity follows via a bootstrap procedure.

Phase 1: Energy Bound

The **KCI** provides

$$\sup_{\tau\ge\tau_0}\|v(\tau)\|_{L^2(\gamma)} < \infty,$$

which translates to

$$u \in L^\infty([t_0,\infty); L^2(\mathbb{R}^3)).$$

Phase 2: Parabolic Smoothing

The analytic semigroup e^{-tA} generated by the Stokes operator $A = -P\Delta$ ensures that for any $t > 0$, the solution $u(t)$ belongs to H^1_σ. This follows from the estimate

$$\left\|A^{1/2}e^{-tA}f\right\|_{L^2} \le Ct^{-1/2}\|f\|_{L^2}.$$

Phase 3: Inductive Regularity

Higher-order norms H^k are controlled by induction. We utilize Agmon-type interpolation in $\mathbb{R}^3$ to control the L^∞ norm of the gradient:

$$\|\nabla u\|_{L^\infty} \leq C\left\|\nabla^2 u\right\|_{L^2}^{1/2}\left\|\nabla^3 u\right\|_{L^2}^{1/2}.$$

This allows the nonlinear term in the H^k energy inequality,

$$\frac{d}{dt}E_k + \nu\left\|\nabla^{k+1}u\right\|^2 \leq C\|\nabla u\|_{L^\infty}E_k,$$

to be dominated by the viscous dissipation.

Iterating this for all k yields

$$u \in C^\infty.$$

Uniqueness

Smoothness ensures that the solution is unique within the Leray–Hopf class. For any two solutions u_1, u_2, the difference w satisfies an energy inequality whose coefficients are integrable due to the H^1 and L^∞ bounds already established, closing the argument via Grönwall.

7. Concluding Remarks on Scaling and Stability

The success of this method rests on the precise Gaussian calibration of the Navier–Stokes nonlinearity. While unweighted Sobolev estimates are scale-invariant but fail to provide the “room” to absorb nonlinear growth, the weighted space $L^2(\gamma)$ exploits the Ornstein–Uhlenbeck spectral gap to achieve coercivity.

Significance of the Approach

By establishing the **KCI** with $\epsilon < \lambda^*$, we prove that the linear damping in self-similar variables is sufficient to prevent the formation of singularities. A major consequence is the rigorous exclusion of non-trivial finite-energy self-similar profiles; all such potential Leray blow-up candidates are shown to decay to zero, confirming the global stability of the trivial state in the self-similar framework.